PHYS 798C Fall 2025
Lecture 21 Summary

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In Lecture 13 we considered tunneling of quasiparticles between superconductors, and between normal metals and superconductors. This is only part of the story. We now consider tunneling of Cooper pairs between superconductors.

I. THE JOSEPHSON EFFECT

Josephson tunneling of Cooper pairs takes place between two superconductors separated by a "weak link" in which the order parameter is suppressed. Many varieties of weak links exist, but it is easiest to do the calculation for the case of an insulating barrier between the two superconductors (SIS tunneling). The superconductors have macroscopic quantum wavefunctions given by $\Psi_1 = \sqrt{n_1^*}e^{i\theta_1}$ and $\Psi_2 = \sqrt{n_2^*}e^{i\theta_2}$, and the barrier between them has thickness $2a$.
Start with the time-independent Schrodinger equation for Ψ and the equation for the the current: $\Lambda\vec{J}_s(\vec{r},t) = \frac{\hbar}{e^*}\vec{\nabla}\theta - \vec{A}(\vec{r},t)$ with $\Psi(\vec{r},t) = \sqrt{n^*}e^{i\theta(\vec{r},t)}$.
Make two assumptions:
1) The junction area is small so that the current density is uniform across the junction. In other words the cross-sectional area is small compared to λ_{eff}^2. This makes the problem one-dimensional.
2) Take the magnetic $\vec{A} = 0$ and electric $\phi = 0$ fields to be zero. These will be added back in later.

One now has a standard quantum barrier tunneling problem (Homework 2) with solution
$J_s = -\frac{e^*\hbar}{m^*\zeta}\frac{\sqrt{n_1^* n_2^*}}{2\sinh(2a/\zeta)}\sin(\theta_1 - \theta_2) \equiv J_c\sin(\theta_1 - \theta_2)$,
where $\zeta^2 = \frac{\hbar^2}{2m^*(V_0 - E)}$, and $V_0 - E$ is the barrier height. The prefactor is the critical current density of the junction,
$J_c = \frac{e^*\hbar}{m^*\zeta}\frac{\sqrt{n_1^* n_2^*}}{2\sinh(2a/\zeta)}$ and depends on the geometry of the junction as well as the superconductors involved.
For thick insulators this reduces to,
$J_c = \frac{e^*\hbar}{m^*\zeta}\frac{\sqrt{n_1^* n_2^*}}{2}e^{-2a/\zeta}$.
The exponential dependence of critical current on barrier thickness and height makes it extremely difficult to make large numbers of Josephson junctions with identical properties, a necessary requirement for applications such as large-scale computing.
The critical current will have the same temperature dependence as the superfluid: $J_c(T) \propto n_s(T)$ as $T \to T_c$.
Note that a supercurrent will flow through the insulating barrier in the absence of a potential difference across the junction. The magnitude of the current depends sinusoidally on the difference in phases of the MQWF on either side of the barrier. The resulting current can be positive, negative, or zero.
This result also suggests that the Josephson current-phase relationship is $\sin(\theta_1 - \theta_2)$, which is found to be correct in many low-T_c Josephson junctions (JJs), but deviations from this simple sinusoidal dependence are seen in disordered d-wave JJs, as illustrated on the class web site.

Bardeen and Josephson had a fundamental disagreement about Cooper pair tunneling through an insulating barrier. Bardeen (using the BCS k-space picture) believed that since $V_{k,k'} = 0$ in the insulator, there could be no support for Cooper pairs and therefore such tunneling was incoherent. If the tunneling probability for a single particle is t (with $t << 1$) then the tunneling probability for a Cooper pair is t^2, and therefore will be swamped by quasiparticle tunneling. Josephson was following the work on generalization of BCS to real space, where it was predicted that the pair potential $\Delta(r)$ was non-zero in the insulator. Therefore he wrote down a tunneling Hamiltonian in which pair tunneling swamped the quasiparticle tunneling. Josephson turned out to be correct, and he won the Nobel prize in physics the year after BCS did for their theory of superconductivity.

II. THE JOSEPHSON JUNCTION IN A MAGNETIC FIELD

At this point we have the dc Josephson effect, which is a spontaneous Cooper pair curent that flows between two superconductors separated by a weak link as $J_s = J_c\sin(\theta_1 - \theta_2)$, where J_s is the super-

current density, J_c is the critical current density (dependent on the barrier height and thickness), and $\theta_1 - \theta_2$ is the difference in phases of the macroscopic quantum wave functions in the two superconductors. Now we wish to include the effect of a magnetic field on the Josephson junction. We shall assume that the superconducting banks remain in the Meissner state and look at the effects of the field on the junction properties. To do this, we appeal to the gauge invariance of the observables, namely $|\Psi(r,t)|^2$ and $J_s = \frac{q^* n^*}{m^*}(\hbar\vec{\bigtriangledown}\theta - q^*\vec{A})$, and demand that their values not depend on a choice of gauge for $\vec{A}$ and $\vec{B}$. A new gauge can be created as $\vec{A'} = \vec{A} + \vec{\bigtriangledown}\chi(r)$, where $\chi(r)$ is an arbitrary scalar function of position. This will leave J_s and $|\Psi(r,t)|^2$ invariant if we also modify the phase of the macroscopic quantum wavefunction as $\theta' = \theta + \frac{q^*}{\hbar}\chi(r)$. Using $q^* = -2e$, we have a new phase difference on the junction $\gamma = \theta_1' - \theta_2' - \frac{2\pi}{\Phi_0}(\chi_1 - \chi_2)$. Writing the difference in χ as the line integral of $\vec{\bigtriangledown}\chi(r)$, we get this expression for the gauge-invariant phase difference γ as,
$\gamma = \theta_1 - \theta_2 - \frac{2\pi}{\Phi_0}\int_1^2 \vec{A}\cdot d\vec{l}$. One can show that the change in gauge introduced above leaves this quantity unchanged.
Now we have the result (DC Josephson effect) that,

$$J_s = J_c \sin(\gamma), \text{with}$$
$$\gamma = \theta_1 - \theta_2 - \frac{2\pi}{\Phi_0}\int_1^2 \vec{A}\cdot d\vec{l}$$

as a more complete expression for the dc Josephson effect. We can see that an applied magnetic field $\vec{B} = \vec{\nabla}\times\vec{A}$ has the ability to modify the supercurrent flowing through the junction.

III. THE AC JOSEPHSON EFFECT

We wish to understand the dynamics of a Josephson junction. If a supercurrent does not cause the phase difference γ to "wind", then what does?
Take the time derivative of the gauge invariant phase difference,
$\frac{\partial\gamma}{\partial t} = \frac{\partial\theta_1}{\partial t} - \frac{\partial\theta_2}{\partial t} - \frac{2\pi}{\Phi_0}\frac{\partial}{\partial t}\int_1^2 \vec{A}\cdot d\vec{l}$
Back in HW2 you derived an expression for the dynamics of the phase of the macroscopic quantum wavefunction $\Psi(r,t) = \sqrt{n^*}e^{i\theta(r,t)}$, where n^* is assumed independent of space and time, as,
$-\hbar\frac{\partial\theta}{\partial t} = \frac{1}{2n^*}\Lambda J_s^2 + q^*\phi$, where ϕ is the electrostatic potential.
Using this in the expression for $\frac{\partial\gamma}{\partial t}$, assuming that the kinetic energy of the supercurrent is continuous across the junction (i.e. $\Lambda_1 J_s^2(a)/2n_1^* = \Lambda_2 J_s^2(-a)/2n_2^*$. Note that $\Lambda J_s^2/2n^*$ reduces to $m^* v_s^2/2$), and that the difference in scalar potential can be written as the line integral of the gradient, we arrive at,
$\frac{\partial\gamma}{\partial t} = \frac{2\pi}{\Phi_0}\int_1^2\left(-\vec{\bigtriangledown}\phi - \frac{\partial\vec{A}}{\partial t}\right)\cdot d\vec{l}$.
The quantity in parentheses is the total electric field, that due to both scalar and vector sources. Hence we have
$\frac{\partial\gamma}{\partial t} = \frac{2\pi}{\Phi_0}\int_1^2 \vec{E}\cdot d\vec{l} \equiv \frac{2\pi}{\Phi_0}\Delta V$.
This integral is just the (full time-dependent) potential difference ΔV between the superconductors, yielding the famous AC Josephson effect expression:

$$\boxed{\frac{\partial\gamma}{\partial t} = \frac{2\pi}{\Phi_0}\Delta V}$$

Hence, by applying a dc potential difference across the junction you can cause the gauge-invariant phase difference to "wind".

IV. CIRCUIT MODEL OF A JOSEPHSON JUNCTION

One can look at a Josephson junction as a lumped circuit element. By integrating the current density over the entire junction one can relate the total current through the device to the gauge-invariant phase difference (GIPD) across the device: $I = I_c\sin(\gamma)$. In the case of a voltage drop V across the junction, the GIPD will wind as

$$\boxed{V = \frac{\Phi_0}{2\pi}\frac{d\gamma}{dt}}$$

.

Suppose a static dc voltage V_{dc} is applied to the junction. The GIPD can be found from integration: $\gamma(t) = \gamma(0) + \frac{2\pi}{\Phi_0} V_{dc} t$. This leads to an alternating current through the junction, given by $I = I_c \sin(2\pi f_J t + \gamma(0))$. The Josephson frequency is $f_J = \frac{V_{dc}}{h/2e} = 483.6$ (THz/V) $V_{dc} = 483.6$ (MHz/μV) V_{dc}. The JJ acts as a very precise voltage-to-frequency transducer and vice versa.

The NIST (and world) voltage standard is based on generating a precise mm-wave signal (at about 90 GHz) and shining it on a series array of Josephson junctions that are designed to yield a total dc voltage drop of precisely 1 volt.

Going the other way, one can use intrinsic Josephson junctions that occur in layered high-T$_c$ cuprates (like Bi-Sr-Ca-Cu-O, aka Bi2212), biased by a dc voltage, to create a coherent mm-wave and THz source. The output frequency can be tuned by about 10 to 20% by altering the dc voltage. In principle the output power should scale with the number of junction layers squared, and it does. However as the stacks of JJs grow thicker they fail to operate properly due to internal heating and other sources of nonlinearity. These applications are illustrated on the class web site.

A more complete lumped circuit treatment of a Josephson junction includes the shunting capacitance C, which carries a displacement current, and the quasiparticle tunneling current, which can exist whenever there is a finite voltage across the junction (Lecture 13). The quasiparticle tunnel current is modeled as a nonlinear (voltage-dependent) resistor R. This circuit approximation to the high frequency response of a Jospehson junction is known as the Resistivity and Capacitively Shunted Junction model, or RCSJ-model. It is generally quite successful at describing the three currents that flow in parallel through real-life Josephson junctions at finite frequency.